

Definition of fractal measures arising from fractional calculus

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The sets and curves of fractional dimension have been constructed and found to be useful at number of places in science [1]. They are used to model various irregular phenomena. It is wellknown that the usual calculus is inadequate to handle such structures and processes. Therefore a new calculus should be developed which incorporates fractals naturally. Fractional calculus, which is a branch of mathematics dealing with derivatives and integrals of fractional order, is one such candidate. The relation between ordinary calculus and measures on \mathbb{R}^n is wellknown. For example, an n -fold integration gives an n -dimensional volume. Also, the solution of $df/dx = 1_{[0,x]}$, where $1_{[0,x]}$ is an indicator function of $[0, x]$, gives length of the interval $[0, x]$ [2]. The aim of this paper is to arrive at a definition of a fractal measure using the concepts from the fractional calculus. Here we shall restrict ourselves to simple subsets of $[0, 1]$ and more rigorous treatment will be given elsewhere.

We first define a differential of fractional order α ($0 \leq \alpha \leq 1$) as follows: $d^\alpha x = d^{-\alpha} 1_{dx}(x)/dx^{-\alpha}$ where

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)}{(x-y)^{q+1}} dy, \quad \text{for } q < 0, \quad (1)$$

is the Riemann-Liouville fractional integral [3]. Now we define a "fractal integral" by ${}_a D_b^{-\alpha} f(x) = \int_a^b f(x) d^\alpha x$, written in discrete form as, ${}_a D_b^{-\alpha} f(x) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(x_i^*) d^{-\alpha} 1_{dx_i} / [d(x_{i+1} - x_i)]^{-\alpha}$, where $[x_i, x_{i+1}]$, $i = 0, \dots, N-1$, $x_0 = a$ and $x_N = b$, provide a partition of the interval $[a, b]$ and x_i^* is some suitably chosen point of the subinterval $[x_i, x_{i+1}]$. We now define the fractional measure of a subset $A \cap [0, x]$ (assuming it to be measurable) as $\mathcal{F}^\alpha(A \cap [0, x]) = {}_0 D_x^{-\alpha} 1_A(x)$. Consider an example of a one-third Cantor set C with dimension $d = \ln(2)/\ln(3)$. For this set \mathcal{F} can be written as $\mathcal{F}^\alpha(C) = {}_0 D_1^{-\alpha} 1_C(x)$. Now we choose x_i^* to be such that $1_C(x_i^*)$ is the maximum in that interval, then

$$\mathcal{F}^\alpha(C) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)}, \quad (2)$$

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where F_C^i is a flag function which is 1 if a point of set C belongs to the interval $[x_i, x_{i+1}]$ and zero otherwise. Clearly this measure is infinite if $\alpha < d$ and zero if $\alpha > d$. At $\alpha = d$ we have $\mathcal{F}^d(C) = 1/\Gamma(d+1)$ whereas the Hausdorff measure [1] $\mathcal{H}^d(C) = [\Gamma(1/2)]^d/\Gamma(1+d/2)$.

Recently, a new quantity viz. local fractional derivative (LFD), was defined [4] as

$$\mathcal{D}^q f(y) = \lim_{x \rightarrow y} \frac{d^q[f(x) - f(y)]}{[d(x - y)]^q} \quad 0 < q \leq 1, \quad x > y, \quad (3)$$

where the RHS uses Riemann-Liouville fractional derivative [3] given by

$$\frac{d^q f(x)}{[d(x - a)]^q} = \frac{1}{\Gamma(1 - q)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x - y)^q} dy \quad \text{for } 0 < q < 1. \quad (4)$$

We also introduced [5] local fractional differential equations which involve LFDs. A solution of the equation [5] $\mathcal{D}^\alpha f(x) = 1_C(x)$ turns out to be equation (2) implying $f(x) = \mathcal{F}^\alpha(C \cap [0, x])$ [2]. This generalizes the fact that the solution of $f'(x) = 1_{[0, x]}$ is the length of the interval $[0, x]$.

A local fractional diffusion equation given by $\mathcal{D}_t^\alpha W(x, t) = (1_C(t)/2)(\partial^2 W(x, t)/\partial x^2)$ (compare Ref. [5]), where $W(x, t)$ is a probability density for finding a particle in neighbourhood of x at time t , has a solution given by [5], for $W(x, 0) = \delta(x)$,

$$W(x, t) = \frac{1}{\sqrt{2\pi\mathcal{F}(C \cap [0, t])}} \exp\left(\frac{-x^2}{2\mathcal{F}(C \cap [0, t])}\right). \quad (5)$$

The mean square displacement, $\langle x^2 \rangle = 2\mathcal{F}(C \cap [0, t])$, is proportional to t^α . Hence the equation (5) gives a subdiffusive solution.

We have introduced a definition of fractal measures using fractional calculus and shown it to be useful in studying diffusion in fractal time.

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